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$\mathbb{Z}_n$ elliptic Gaudin model with open boundaries

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Abstract: The $\mathbb{Z}_n$ elliptic Gaudin model with integrable boundaries specified by generic *non-diagonal* K-matrices with $n+1$ free boundary parameters is studied. The commuting families of Gaudin operators are diagonalized by the algebraic Bethe ansatz method. The eigenvalues and the corresponding Bethe ansatz equations are obtained.

Keywords: Lattice Integrable Models, Bethe Ansatz.

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Contents

1. Introduction

In the study of one-dimensional many-body systems with long-range interactions Gaudin type models [1] stand out as a particularly important class due to their applications in many branches of physics such as the BCS theory [2] of small metallic grains [3, 4, 5], theoretical nuclear physics [6, 7], quantum chromo-dynamics (QCD) theory [8, 9] and Seiberg-Witten theory of supersymmetric gauge theory [10]. They also provide a powerful way for the construction of integral representations of solutions to the Knizhnik-Zamolodchikov (KZ) equation [11]–[15].

Rational and trigonometric Gaudin magnets have been extensively investigated in the literature. It is well-known that the periodic Gaudin's magnet hamiltonians (or Gaudin operators) can be constructed via the quasi-classical expansion of the transfer matrix (*row-to-row transfer matrix*) of an inhomogeneous spin chain with periodic boundary conditions [16, 17]. Gaudin models with non-trivial boundary conditions can be in principle treated by means of the Sklyanin's boundary inverse scattering method [18]. In this case the quasi-classical expansion of the corresponding boundary transfer matrix (*double-row transfer matrix*) produces *generalized* Gaudin hamiltonians with boundaries specified by certain K-matrices [14, 19]. In particular, twisted boundary conditions and open boundary conditions associated with *diagonal* K-matrices give rise to Gaudin magnets in non-uniform local magnetic fields [14] and interacting electron pairs with certain non-uniform long-range coupling strengths [4, 20, 21, 22, 19]. Very recently, the XXZ Gaudin model with generic integrable boundaries specified by generic *non-diagonal* K-matrices [23, 24] was solved by the algebraic Bethe ansatz method [25].

So far, study of Gaudin type models via the *algebraic Bethe ansatz method* is quite advanced for rational and trigonometric interaction cases. On the other hand, the elliptic interaction case is rather less developed. We are aware of only two works [17, 15] where the XYZ Gaudin model and its face-type counterpart were respectively constructed and solved. In this paper we will solve, using the algebraic Bethe ansatz method, the most *generic* case– the $\mathbb{Z}_n$ elliptic Gaudin magnets with open boundary conditions specified by the generic *non-diagonal* K-matrices given in [26, 27], which contain $n+1$ free boundary parameters. In section 3, we construct the elliptic Gaudin operators associated with the generic boundary K-matrices. The commutativity of these operators follows from the standard procedure [16, 14, 19] specializing to the inhomogeneous $\mathbb{Z}_n$ Belavin model with open boundaries [28, 29], thus ensuring the integrability of the $\mathbb{Z}_n$ elliptic Gaudin magnets. In section 4, we diagonalize the Gaudin operators simultaneously by means of the algebraic Bethe ansatz method. This constitutes the main new result in this paper. The diagonalization is achieved by means of the technique of the "vertex-face" transformation. In section 5, we conclude this paper with a summary and comments.

2. Preliminaries: the inhomogeneous $\mathbb{Z}_n$ Belavin model with open boundaries

Let us fix a positive integer $n \geq 2$, a complex number τ such that $Im(\tau) > 0$ and a generic complex number w. Introduce the following elliptic functions

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(u,\tau) = \sum_{m=-\infty}^{\infty} exp\left\{\sqrt{-1}\pi\left[(m+a)^2\tau + 2(m+a)(u+b)\right]\right\}, \tag{2.1}$$

$$\theta^{(j)}(u) = \theta\begin{bmatrix} \frac{1}{2}-\frac{j}{n} \\ \frac{1}{2} \end{bmatrix}(u,n\tau), \qquad \sigma(u) = \theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(u,\tau), \tag{2.2}$$

$$\zeta(u) = \frac{\partial}{\partial u}\{\ln\sigma(u)\}. \tag{2.3}$$

Among them the σ-function[1] satisfies the following identity:

$$\sigma(u+x)\sigma(u-x)\sigma(v+y)\sigma(v-y) - \sigma(u+y)\sigma(u-y)\sigma(v+x)\sigma(v-x) =$$
$$= \sigma(u+v)\sigma(u-v)\sigma(x+y)\sigma(x-y).$$

Let g, h, be $n \times n$ matrices with the elements

$$h_{ij} = \delta_{i+1\,j}, \qquad g_{ij} = \omega^i\delta_{i\,j}, \qquad \text{with } \omega = e^{\frac{2\pi\sqrt{-1}}{n}}, \qquad i,j \in \mathbb{Z}_n.$$

For any $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 \in \mathbb{Z}_n$, one can introduce an associated $n \times n$ matrix I_α defined by

$$I_\alpha = I_{(\alpha_1,\alpha_2)} = g^{\alpha_2}h^{\alpha_1},$$

[1] Our σ-function is the ϑ-function $\vartheta_1(u)$ [30]. It has the following relation with the *Weierstrassian* σ-function if denoted it by $\sigma_w(u)$: $\sigma_w(u) \propto e^{\eta_1 u^2}\sigma(u)$, $\eta_1 = \pi^2(\frac{1}{6} - 4\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}})$ and $q = e^{\sqrt{-1}\tau}$. Consequently, our ζ-function given by (2.3) is different from the *Weierstrassian* ζ-function by an additional term $-2\eta_1 u$.

and an elliptic function $\sigma_\alpha(u)$ given by

$$\sigma_\alpha(u) = \theta \begin{bmatrix} \frac{1}{2} + \frac{\alpha_1}{n} \\ \frac{1}{2} + \frac{\alpha_2}{n} \end{bmatrix} (u, \tau), \qquad \text{and} \qquad \sigma_{(0,0)}(u) = \sigma(u).$$

The $\mathbb{Z}_n$ Belavin R-matrix is given by [28]

$$R^B(u) = \frac{\sigma(w)}{\sigma(u+w)} \sum_{\alpha \in \mathbb{Z}_n^2} \frac{\sigma_\alpha(u + \frac{w}{n})}{n\sigma_\alpha(\frac{w}{n})} I_\alpha \otimes I_\alpha^{-1}. \tag{2.4}$$

The R-matrix satisfies the quantum Yang-Baxter equation (QYBE)

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2), \tag{2.5}$$

and the properties [31],

$$\text{Unitarity:} \qquad R^B_{12}(u)R^B_{21}(-u) = \text{id}, \tag{2.6}$$

$$\text{Crossing-unitarity:} \qquad (R^B)^{t_2}_{21}(-u - nw)(R^B)^{t_2}_{12}(u) = \frac{\sigma(u)\sigma(u+nw)}{\sigma(u+w)\sigma(u+nw-w)} \text{id}, \tag{2.7}$$

$$\text{Quasi-classical property:} \qquad R^B_{12}(u)|_{w \to 0} = \text{id}. \tag{2.8}$$

Here $R^B_{21}(u) = P_{12}R^B_{12}(u)P_{12}$ with P_{12} being the usual permutation operator and t_i denotes the transposition in the i-th space. Here and below we adopt the standard notation: for any matrix $A \in \text{End}(\mathbb{C}^n)$, A_j is an embedding operator in the tensor space $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots$, which acts as A on the j-th space and as an identity on the other factor spaces; $R_{ij}(u)$ is an embedding operator of R-matrix in the tensor space, which acts as an identity on the factor spaces except for the i-th and j-th ones. The quasi-classical properties (2.8) of the R-matrix enables one to introduce an associated $\mathbb{Z}_n$ classical r-matrix $r(u)$ [32]

$$\begin{aligned} R^B(u) &= \text{id} + w\, r(u) + O(w^2), \qquad \text{when } w \longrightarrow 0, \\ r(u) &= \frac{1-n}{n}\zeta(u) + \sum_{\alpha \in \mathbb{Z}_n^2 - (0,0)} \frac{\sigma'(0)\sigma_\alpha(u)}{n\sigma(u)\sigma_\alpha(0)} I_\alpha \otimes I_\alpha^{-1}, \qquad \sigma'(0) = \frac{\partial}{\partial u}\sigma(u)|_{u=0}. \end{aligned} \tag{2.9}$$

In the above equation, the elliptic ζ-function is defined in (2.3).

One introduces the "row-to-row" monodromy matrix $T(u)$ [33], which is an $n \times n$ matrix with elements being operators acting on $(\mathbb{C}^n)^{\otimes N}$

$$T(u) = R^B_{01}(u + z_1)R^B_{02}(u + z_2) \cdots R^B_{0N}(u + z_N). \tag{2.10}$$

Here $\{z_i | i = 1, \ldots, N\}$ are arbitrary free complex parameters which are usually called inhomogeneous parameters. With the help of the QYBE (2.5), one can show that $T(u)$ satisfies the so-called "RLL" relation

$$R^B_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R^B_{12}(u - v). \tag{2.11}$$

An integrable open chain can be constructed as follows [18]. Let us introduce a pair of K-matrices $K^-(u)$ and $K^+(u)$. The former satisfies the reflection equation (RE)

$$R^B_{12}(u_1 - u_2)K^-_1(u_1)R^B_{21}(u_1 + u_2)K^-_2(u_2) = K^-_2(u_2)R^B_{12}(u_1 + u_2)K^-_1(u_1)R^B_{21}(u_1 - u_2), \tag{2.12}$$

and the latter satisfies the dual RE

$$R^B_{12}(u_2-u_1)K^+_1(u_1)R^B_{21}(-u_1-u_2-nw)K^+_2(u_2) = K^+_2(u_2)R^B_{12}(-u_1-u_2-nw)\times \\ \times K^+_1(u_1)R^B_{21}(u_2-u_1)\,. \qquad (2.13)$$

For the models with open boundaries, instead of the standard "row-to-row" monodromy matrix $T(u)$ (2.10), one needs the "double-row" monodromy matrix $\mathbb{T}(u)$

$$\mathbb{T}(u) = T(u)K^-(u)T^{-1}(-u)\,. \qquad (2.14)$$

Using (2.11) and (2.12), one can prove that $\mathbb{T}(u)$ satisfies

$$R^B_{12}(u_1-u_2)\mathbb{T}_1(u_1)R^B_{21}(u_1+u_2)\mathbb{T}_2(u_2) = \mathbb{T}_2(u_2)R^B_{12}(u_1+u_2)\mathbb{T}_1(u_1)R^B_{21}(u_1-u_2)\,. \qquad (2.15)$$

Then the *double-row transfer matrix* of the inhomogeneous $\mathbb{Z}_n$ Belavin model with open boundary is given by

$$\tau(u) = \mathrm{tr}(K^+(u)\mathbb{T}(u))\,. \qquad (2.16)$$

The commutativity of the transfer matrices

$$[\tau(u),\tau(v)] = 0\,, \qquad (2.17)$$

follows as a consequence of (2.5)–(2.7) and (2.12)–(2.13). This ensures the integrability of the inhomogeneous $\mathbb{Z}_n$ Belavin model with open boundary.

3. $\mathbb{Z}_n$ elliptic Gaudin models with generic boundaries

In this paper, we will consider a *non-diagonal* K-matrix $K^-(u)$ which is a solution to the RE (2.12) associated with the $\mathbb{Z}_n$ Belavin R-matrix [26]

$$K^-(u)^s_t = \sum_{i=1}^n \frac{\sigma(\lambda_i+\xi-u)}{\sigma(\lambda_i+\xi+u)}\,\phi^{(s)}_{\lambda,\lambda-w\hat{\imath}}(u)\,\bar{\phi}^{(t)}_{\lambda,\lambda-w\hat{\imath}}(-u)\,. \qquad (3.1)$$

The corresponding *dual* K-matrix $K^+(u)$ which is a solution to the dual RE (2.13) has been obtained in [27]. With a particular choice of the free boundary parameters with respect to $K^-(u)$, we have

$$K^+(u)^s_t = \sum_{i=1}^n \left\{\prod_{k\neq i}\frac{\sigma((\lambda_i-\lambda_k)-w)}{\sigma(\lambda_i-\lambda_k)}\right\}\frac{\sigma(\lambda_i+\bar{\xi}+u+\frac{nw}{2})}{\sigma(\lambda_i+\bar{\xi}-u-\frac{nw}{2})}\phi^{(s)}_{\lambda,\lambda-w\hat{\imath}}(-u)\,\tilde{\phi}^{(t)}_{\lambda,\lambda-w\hat{\imath}}(u)\,. \qquad (3.2)$$

In (3.1) and (3.2), ϕ, $\bar{\phi}$, $\tilde{\phi}$ are intertwiners which will be specified in section 4. We consider the generic $\{\lambda_i\}$ such that $\lambda_i \neq \lambda_j$ (*modulo* $\mathbb{Z}+\tau\mathbb{Z}$) for $i\neq j$. This condition is necessary for the non-singularity of $K^\pm(u)$. It is convenient to introduce a vector $\lambda = \sum_{i=1}^n \lambda_i\epsilon_i$ associated with the boundary parameters $\{\lambda_i\}$, where $\{\epsilon_i,\ i=1,\ldots,n\}$ is the orthonormal basis of the vector space $\mathbb{C}^n$ such that $\langle\epsilon_i,\epsilon_j\rangle = \delta_{ij}$.

As will be seen from the definitions of the intertwiners (4.8), (4.10) and (4.11) specialized to $m=\lambda$, $\phi_{\lambda,\lambda-w\hat{\imath}}(u)$ and $\bar{\phi}_{\lambda,\lambda-w\hat{\imath}}(u)$ do not depend on w but $\tilde{\phi}_{\lambda,\lambda-w\hat{\imath}}(u)$ does.

Consequently, the K-matrix $K^{-}(u)$ does not depend on the crossing parameter w, but $K^{+}(u)$ does. So we use the convention:

$$K(u) = \lim_{w\to 0} K^{-}(u) = K^{-}(u)\,. \tag{3.3}$$

We further restrict the complex parameters ξ and $\bar{\xi}$ to be the same, i.e.

$$\bar{\xi} = \xi\,, \tag{3.4}$$

so that (3.5) below is satisfied. Hence, the K-matrices depend on $n+1$ free parameters $\{\lambda_i | i = 1, \ldots, n\}$ and ξ, which specify the integrable boundary conditions [24]. Moreover, the K-matrices satisfy the following relation thanks to the restriction (3.4)

$$\lim_{w\to 0}\{K^{+}(u)\, K^{-}(u)\} = \lim_{w\to 0}\{K^{+}(u)\}K(u) = \mathrm{id}\,. \tag{3.5}$$

Let us introduce the $\mathbb{Z}_n$ elliptic Gaudin operators $\{H_j | j = 1, 2, \ldots, N\}$ associated with the inhomogeneous $\mathbb{Z}_n$ Belavin model with open boundaries specified by the generic K-matrices (3.1) and (3.2):

$$H_j = \Gamma_j(z_j) + \sum_{k\neq j}^{N} r_{kj}(z_j - z_k) + K_j^{-1}(z_j)\left\{\sum_{k\neq j}^{N} r_{jk}(z_j + z_k)\right\} K_j(z_j)\,, \tag{3.6}$$

where $\Gamma_j(u) = \frac{\partial}{\partial w}\{\bar{K}_j(u)\}|_{w=0} K_j(u)$, $j = 1, \ldots, N$, with $\bar{K}_j(u) = tr_0\left\{K_0^{+}(u) R_{0j}^{B}(2u) P_{0j}\right\}$. Here $\{z_j\}$ are the inhomogeneous parameters of the inhomogeneous $\mathbb{Z}_n$ Belavin model and $r(u)$ is given by (2.9). For a generic choice of the boundary parameters $\{\lambda_1, \ldots, \lambda_n, \xi\}$, $\Gamma_j(u)$ is a *non-diagonal* matrix.

The elliptic Gaudin operators (3.6) are obtained by expanding the double-row transfer matrix (2.16) at the point $u = z_j$ around $w = 0$:

$$\tau(z_j) = \tau(z_j)|_{w=0} + wH_j + O(w^2)\,, \qquad j = 1, \ldots, N\,, \tag{3.7}$$

$$H_j = \frac{\partial}{\partial w}\tau(z_j)|_{w=0}\,. \tag{3.8}$$

The relations (2.8) and (3.5) imply that the first term $\tau(z_j)|_{w=0}$ in the expansion (3.7) is equal to an identity, namely,

$$\tau(z_j)|_{w=0} = \mathrm{id}\,. \tag{3.9}$$

Then the commutativity of the transfer matrices $\{\tau(z_j)\}$ (2.17) for a generic w implies

$$[H_j, H_k] = 0\,, \qquad i, j = 1, \ldots, N.\,. \tag{3.10}$$

Thus the elliptic Gaudin system defined by (3.6) is integrable. Moreover, the fact that the Gaudin operators $\{H_j\}$ (3.6) can be expressed in terms of the transfer matrix of the inhomogeneous $\mathbb{Z}_n$ Belavin model with open boundary enables us to exactly diagonalize the operators by the algebraic Bethe ansatz method with the help of the "vertex-face" correspondence technique, as will be shown in the next section. The aim of this paper is to diagonalize the elliptic Gaudin operators H_j, $j = 1, \ldots, N$ (3.6) simultaneously.

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4. Eigenvalues and Bethe ansatz equations

4.1 $A^{(1)}_{n-1}$ SOS R-matrix and face-vertex correspondence

The A_{n-1} simple roots $\{\alpha_i\}$ can be expressed in terms of the orthonormal basis $\{\epsilon_i\}$ as:

$$\alpha_i = \epsilon_i - \epsilon_{i+1}\,, \qquad i = 1, \ldots, n-1\,,$$

and the fundamental weights $\{\Lambda_i \mid i = 1, \ldots, n-1\}$ satisfying $\langle \Lambda_i,\ \alpha_j\rangle = \delta_{ij}$ are given by

$$\Lambda_i = \sum_{k=1}^{i} \epsilon_k - \frac{i}{n}\sum_{k=1}^{n} \epsilon_k\,.$$

Set

$$\hat{\imath} = \epsilon_i - \overline{\epsilon}\,, \qquad \overline{\epsilon} = \frac{1}{n}\sum_{k=1}^{n}\epsilon_k\,, \qquad i = 1, \ldots, n\,, \qquad \text{then } \sum_{i=1}^{n} \hat{\imath} = 0\,. \tag{4.1}$$

For each dominant weight $\Lambda = \sum_{i=1}^{n-1} a_i \Lambda_i$, $a_i \in \mathbb{Z}^+$ (the set of non-negative integers), there exists an irreducible highest weight finite-dimensional representation V_Λ of A_{n-1} with the highest vector $|\Lambda\rangle$. For example the fundamental vector representation is V_{Λ_1}.

Let $\mathfrak{h}$ be the Cartan subalgebra of A_{n-1} and $\mathfrak{h}^*$ be its dual. A finite dimensional diagonalizable $\mathfrak{h}$-module is a complex finite dimensional vector space W with a weight decomposition $W = \oplus_{\mu\in\mathfrak{h}^*} W[\mu]$, so that $\mathfrak{h}$ acts on $W[\mu]$ by $x\,v = \mu(x)\,v$, $(x \in \mathfrak{h},\ v \in W[\mu])$. For example, the fundamental vector representation $V_{\Lambda_1} = \mathbb{C}^n$, the non-zero weight spaces $W[\hat{\imath}] = \mathbb{C}\epsilon_i\,,\ i = 1, \ldots, n$.

For a generic $m \in \mathbb{C}^n$, define

$$m_i = \langle m, \epsilon_i\rangle\,, \qquad m_{ij} = m_i - m_j = \langle m, \epsilon_i - \epsilon_j\rangle\,, \qquad i, j = 1, \ldots, n\,. \tag{4.2}$$

Let $R(u, m) \in End(\mathbb{C}^n \otimes \mathbb{C}^n)$ be the R-matrix of the $A^{(1)}_{n-1}$ SOS model [34] given by

$$R(u,m) = \sum_{i=1}^{n} R^{ii}_{ii}(u,m) E_{ii}\otimes E_{ii} + \sum_{i\neq j}\left\{R^{ij}_{ij}(u,m)E_{ii}\otimes E_{jj} + R^{ji}_{ij}(u,m)E_{ji}\otimes E_{ij}\right\}, \tag{4.3}$$

where E_{ij} is the matrix with elements $(E_{ij})^l_k = \delta_{jk}\delta_{il}$. The coefficient functions are

$$R^{ii}_{ii}(u,m) = 1\,, \qquad R^{ij}_{ij}(u,m) = \frac{\sigma(u)\sigma(m_{ij}-w)}{\sigma(u+w)\sigma(m_{ij})}\,, \quad i \neq j\,, \tag{4.4}$$

$$R^{j\,i}_{ij}(u,m) = \frac{\sigma(w)\sigma(u+m_{ij})}{\sigma(u+w)\sigma(m_{ij})}\,, \quad i \neq j\,, \tag{4.5}$$

and m_{ij} are defined in (4.2). The R-matrix satisfies the dynamical (modified) QYBE

$$\begin{aligned} &R_{12}(u_1-u_2, m-wh^{(3)})R_{13}(u_1-u_3,m)R_{23}(u_2-u_3,m-wh^{(1)}) = \\ &\quad = R_{23}(u_2-u_3,m)R_{13}(u_1-u_3,m-wh^{(2)})R_{12}(u_1-u_2,m)\,, \end{aligned} \tag{4.6}$$

and the quasi-classical property

$$R(u,m)|_{w\to 0} = \text{id}\,. \tag{4.7}$$

We adopt the notation: $R_{12}(u,m-wh^{(3)})$ acts on a tensor $v_1\otimes v_2\otimes v_3$ as $R(u,m-w\mu)\otimes id$ if $v_3\in W[\mu]$. Moreover, the R-matrix satisfies the unitarity and the modified crossing-unitarity relation [34].

Let us introduce an intertwiner—an n-component column vector $\phi_{m,m-w\hat{\jmath}}(u)$ whose k-th element is

$$\phi^{(k)}_{m,m-w\hat{\jmath}}(u)=\theta^{(k)}(u+nm_j)\,. \tag{4.8}$$

Using the intertwining vector, one derives the following face-vertex correspondence relation [34]

$$\begin{aligned}R^B_{12}(u_1-u_2)\,\phi_{m,m-w\hat{\imath}}(u_1)\otimes\phi_{m-w\hat{\imath},m-w(\hat{\imath}+\hat{\jmath})}(u_2)&=\\ =\sum_{k,l}R(u_1-u_2,m)^{kl}_{ij}\,\phi_{m-w\hat{l},m-w(\hat{l}+\hat{k})}(u_1)\otimes\phi_{m,m-w\hat{l}}(u_2)\,.\end{aligned} \tag{4.9}$$

Then the QYBE of $\mathbb{Z}_n$ Belavin's R-matrix $R^B(u)$ (2.5) is equivalent to the dynamical Yang-Baxter equation of $A^{(1)}_{n-1}$ SOS R-matrix $R(u,m)$ (4.6). For a generic m, we may introduce other types of intertwiners $\bar{\phi}$, $\tilde{\phi}$ satisfying the conditions,

$$\sum_{k=1}^{n}\bar{\phi}^{(k)}_{m,m-w\hat{\mu}}(u)\;\phi^{(k)}_{m,m-w\hat{\nu}}(u)\;=\;\delta_{\mu\nu}\,, \tag{4.10}$$

$$\sum_{k=1}^{n}\tilde{\phi}^{(k)}_{m+w\hat{\mu},m}(u)\;\phi^{(k)}_{m+w\hat{\nu},m}(u)\;=\;\delta_{\mu\nu}\,, \tag{4.11}$$

from which one derives the relations,

$$\sum_{\mu=1}^{n}\bar{\phi}^{(i)}_{m,m-w\hat{\mu}}(u)\;\phi^{(j)}_{m,m-w\hat{\mu}}(u)\;=\;\delta_{ij}\,, \tag{4.12}$$

$$\sum_{\mu=1}^{n}\tilde{\phi}^{(i)}_{m+w\hat{\mu},m}(u)\;\phi^{(j)}_{m+w\hat{\mu},m}(u)\;=\;\delta_{ij}\,. \tag{4.13}$$

With the help of (4.10)–(4.13), we obtain the following relations from the face-vertex correspondence relation (4.9):

$$\begin{aligned}\Big(\tilde{\phi}_{m+w\hat{k},m}(u_1)\otimes\mathrm{id}\Big)\,R^B_{12}(u_1-u_2)\,\Big(\mathrm{id}\otimes\phi_{m+w\hat{\jmath},m}(u_2)\Big)&=\\ =\sum_{i,l}R(u_1-u_2,m)^{kl}_{ij}\,\tilde{\phi}_{m+w(\hat{\imath}+\hat{\jmath}),m+w\hat{\jmath}}(u_1)\otimes\phi_{m+w(\hat{k}+\hat{l}),m+w\hat{k}}(u_2)\,,\end{aligned} \tag{4.14}$$

$$\begin{aligned}\Big(\tilde{\phi}_{m+w\hat{k},m}(u_1)\otimes\tilde{\phi}_{m+w(\hat{k}+\hat{l}),m+w\hat{k}}(u_2)\Big)\,R^B_{12}(u_1-u_2)&=\\ =\sum_{i,j}R(u_1-u_2,m)^{kl}_{ij}\,\tilde{\phi}_{m+w(\hat{\imath}+\hat{\jmath}),m+w\hat{\jmath}}(u_1)\otimes\tilde{\phi}_{m+w\hat{\jmath},m}(u_2)\,,\end{aligned} \tag{4.15}$$

$$\begin{aligned}\Big(\mathrm{id}\otimes\bar{\phi}_{m,m-w\hat{l}}(u_2)\Big)\,R^B_{12}(u_1-u_2)\,\Big(\phi_{m,m-w\hat{\imath}}(u_1)\otimes\mathrm{id}\Big)&=\\ =\sum_{k,j}R(u_1-u_2,m)^{kl}_{ij}\,\phi_{m-w\hat{l},m-w(\hat{k}+\hat{l})}(u_1)\otimes\bar{\phi}_{m-w\hat{\imath},m-w(\hat{\imath}+\hat{\jmath})}(u_2)\,,\end{aligned} \tag{4.16}$$

$$\left(\bar{\phi}_{m-w\hat{l},m-w(\hat{k}+\hat{l})}(u_1)\otimes\bar{\phi}_{m,m-w\hat{l}}(u_2)\right)R^B_{12}(u_1-u_2)=$$
$$=\sum_{i,j}R(u_1-u_2,m)^{kl}_{ij}\,\bar{\phi}_{m,m-w\hat{\imath}}(u_1)\otimes\bar{\phi}_{m-w\hat{\imath},m-w(\hat{\imath}+\hat{\jmath})}(u_2)\,. \tag{4.17}$$

The face-vertex correspondence relations (4.9) and (4.14)–(4.17) will play an important role in *translating* formulas in the "vertex form" into those in the "face form" so that the algebraic Bethe ansatz method can be applied to diagonalize the transfer matrix.

Corresponding to the vertex type K-matrices (3.1) and (3.2), one introduces the following face type K-matrices $\mathcal{K}$ and $\tilde{\mathcal{K}}$, as in [27]

$$\mathcal{K}(\lambda|u)^j_i=\sum_{s,t}\tilde{\phi}^{(s)}_{\lambda-w(\hat{\imath}-\hat{\jmath}),\ \lambda-w\hat{\imath}}(u)\,K(u)^s_t\,\phi^{(t)}_{\lambda,\ \lambda-w\hat{\imath}}(-u)\,, \tag{4.18}$$
$$\tilde{\mathcal{K}}(\lambda|u)^j_i=\sum_{s,t}\bar{\phi}^{(s)}_{\lambda,\ \lambda-w\hat{\jmath}}(-u)\,\tilde{K}(u)^s_t\,\phi^{(t)}_{\lambda-w(\hat{\jmath}-\hat{\imath}),\ \lambda-w\hat{\jmath}}(u)\,. \tag{4.19}$$

Through straightforward calculations, we find the face type K-matrices *simultaneously* have *diagonal* forms[2]

$$\mathcal{K}(\lambda|u)^j_i=\delta^j_i\,k(\lambda|u;\xi)_i\,,\qquad\tilde{\mathcal{K}}(\lambda|u)^j_i=\delta^j_i\,\tilde{k}(\lambda|u)_i\,, \tag{4.20}$$

where functions $k(\lambda|u;\xi)_i$, $\tilde{k}(\lambda|u)_i$ are given by

$$k(\lambda|u;\xi)_i=\frac{\sigma(\lambda_i+\xi-u)}{\sigma(\lambda_i+\xi+u)}\,, \tag{4.21}$$
$$\tilde{k}(\lambda|u)_i=\left\{\prod_{k\neq i,k=1}^{n}\frac{\sigma(\lambda_{ik}-w)}{\sigma(\lambda_{ik})}\right\}\frac{\sigma(\lambda_i+\bar{\xi}+u+\frac{nw}{2})}{\sigma(\lambda_i+\bar{\xi}-u-\frac{nw}{2})}\,. \tag{4.22}$$

Moreover, one can check that the matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ satisfy the SOS type reflection equation and its dual, respectively [27]. Although the K-matrices $K^{\pm}(u)$ given by (3.1) and (3.2) are generally non-diagonal (in the vertex form), after the face-vertex transformations (4.18) and (4.19), the face type counterparts $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ *simultaneously* become diagonal. This fact enables the authors to apply the generalized algebraic Bethe ansatz method developed in [29] for SOS type integrable models to diagonalize the transfer matrix $\tau(u)$ (2.16).

4.2 Algebraic Bethe ansatz

By means of (4.12), (4.13), (4.19) and (4.20), the transfer matrix $\tau(u)$ (2.16) can be recast into the following face type form:

$$\tau(u)=\mathrm{tr}(K^{+}(u)\mathbb{T}(u))$$

[2] As will be seen below (4.53), the spectral parameter u and the boundary parameter ξ of the reduced double-row monodromy matrices constructed from $\mathcal{K}(\lambda|u)$ will be shifted in each step of the nested Bethe ansatz procedure [29]. Therefore, it is convenient to specify the dependence on the boundary parameter ξ of $\mathcal{K}(\lambda|u)$ in addition to the spectral parameter u.

$$
\begin{aligned}
&= \sum_{\mu,\nu} \mathrm{tr}\Big(K^+(u)\,\phi_{\lambda-w(\hat\mu-\hat\nu),\lambda-w\hat\mu}(u)\,\tilde\phi_{\lambda-w(\hat\mu-\hat\nu),\lambda-w\hat\mu}(u)\times \\
&\qquad\qquad \times \mathbb{T}(u)\phi_{\lambda,\lambda-w\hat\mu}(-u)\bar\phi_{\lambda,\lambda-w\hat\mu}(-u)\Big) \\
&= \sum_{\mu,\nu} \bar\phi_{\lambda,\lambda-w\hat\mu}(-u)K^+(u)\,\phi_{\lambda-w(\hat\mu-\hat\nu),\lambda-w\hat\mu}(u)\;\tilde\phi_{\lambda-w(\hat\mu-\hat\nu),\lambda-w\hat\mu}(u)\;\mathbb{T}(u)\phi_{\lambda,\lambda-w\hat\mu}(-u) \\
&= \sum_{\mu,\nu} \tilde{\mathcal{K}}(\lambda|u)^{\mu}_{\nu}\,\mathcal{T}(\lambda|u)^{\nu}_{\mu} = \sum_{\mu} \tilde k(\lambda|u)_{\mu}\,\mathcal{T}(\lambda|u)^{\mu}_{\mu}\,.
\end{aligned} \tag{4.23}
$$

Here we have introduced the face-type double-row monodromy matrix $\mathcal{T}(\lambda|u)$,

$$
\begin{aligned}
\mathcal{T}(\lambda|u)^{\nu}_{\mu} &= \tilde\phi_{\lambda-w(\hat\mu-\hat\nu),\lambda-w\hat\mu}(u)\;\mathbb{T}(u)\,\phi_{\lambda,\lambda-w\hat\mu}(-u) \\
&\equiv \sum_{i,j} \tilde\phi^{(j)}_{\lambda-w(\hat\mu-\hat\nu),\lambda-w\hat\mu}(u)\;\mathbb{T}(u)^{j}_{i}\,\phi^{(i)}_{\lambda,\lambda-w\hat\mu}(-u)\,.
\end{aligned} \tag{4.24}
$$

This face-type double-row monodromy matrix can be expressed in terms of the face type R-matrix $R(u,\lambda)$ (4.3) and the K-matrix $\mathcal{K}(\lambda|u)$ (4.18) [29]. Moreover from (2.15), (4.9) and (4.13) one may derive the following exchange relations among $\mathcal{T}(\lambda|u)^{\nu}_{\mu}$:

$$
\begin{aligned}
&\sum_{i_1,i_2}\sum_{j_1,j_2} R(u_1-u_2,\lambda)^{i_0,j_0}_{i_1,j_1}\,\mathcal{T}(\lambda+w(\hat\jmath_1+\hat\imath_2)|u_1)^{i_1}_{i_2}\times \\
&\qquad\qquad \times R(u_1+u_2,\lambda)^{j_1,i_2}_{j_2,i_3}\,\mathcal{T}(\lambda+w(\hat\jmath_3+\hat\imath_3)|u_2)^{j_2}_{j_3} = \\
&= \sum_{i_1,i_2}\sum_{j_1,j_2}\;\mathcal{T}(\lambda+w(\hat\jmath_1+\hat\imath_0)|u_2)^{j_0}_{j_1}\,R(u_1+u_2,\lambda)^{i_0,j_1}_{i_1,j_2}\times \\
&\times\mathcal{T}(\lambda+w(\hat\jmath_2+\hat\imath_2)|u_1)^{i_1}_{i_2}\,R(u_1-u_2,\lambda)^{j_2,i_2}_{j_3,i_3}\,.
\end{aligned} \tag{4.25}
$$

As in [29], we introduce a standard notation for convenience:[3]

$$
\mathcal{A}(\lambda|u) = \mathcal{T}(\lambda|u)^{1}_{1}\,,\qquad \mathcal{B}_i(\lambda|u) = \frac{\mathcal{T}(\lambda|u)^{1}_{i}}{\sigma(\lambda_{i1})}\,,\quad i=2,\dots,n\,, \tag{4.26}
$$

$$
\Delta^{j}_{i}(\lambda|u) = \frac{\sigma(\lambda_{j1}-\delta_{ij}w)}{\sigma(\lambda_{i1})}\Big\{\mathcal{T}(\lambda|u)^{j}_{i} - \delta^{j}_{i}R(2u,\lambda+w\hat 1)^{j\,1}_{1\,j}\,\mathcal{A}(\lambda|u)\Big\}\,,\quad i,j=2,\dots,n\,. \tag{4.27}
$$

From (4.25) one may derive the commutation relations among $\mathcal{A}(\lambda|u)$, $\Delta(\lambda|u)$ and $\mathcal{B}(\lambda|u)$. Here we give those which are relevant for our purpose

$$
\begin{aligned}
&\mathcal{A}(\lambda|u)\mathcal{B}_i(\lambda+w(\hat\imath-\hat 1)|v) = \\
&\quad= \frac{\sigma(u+v)\sigma(u-v-w)}{\sigma(u+v+w)\sigma(u-v)}\mathcal{B}_i(\lambda+w(\hat\imath-\hat 1)|v)\mathcal{A}(\lambda+w(\hat\imath-\hat 1)|u) - \\
&\qquad - \frac{\sigma(w)\sigma(2v)}{\sigma(u-v)\sigma(2v+w)}\frac{\sigma(u-v-\lambda_{1i}+w)}{\sigma(\lambda_{1i}-w)}\,\mathcal{B}_i(\lambda+w(\hat\imath-\hat 1)|u)\mathcal{A}(\lambda+w(\hat\imath-\hat 1)|v) - \\
&\qquad - \frac{\sigma(w)}{\sigma(u+v+w)}\sum_{\alpha=2}^{n}\frac{\sigma(u+v+\lambda_{\alpha 1}+2w)}{\sigma(\lambda_{\alpha 1}+w)}\,\mathcal{B}_\alpha(\lambda+w(\hat\alpha-\hat 1)|u)\Delta^{\alpha}_{i}(\lambda+w(\hat\imath-\hat 1)|v)\,,
\end{aligned}
$$

[3]The scalar factors in the definitions of the operators $\mathcal{B}(\lambda|u)$ and $\Delta(\lambda|u)$ are to make the relevant commutation relations as concise as (4.28)–(4.30).

$$
\begin{aligned}
&\Delta_i^k(\lambda|u)\mathcal{B}_j(\lambda+w(\hat{\jmath}-\hat{1})|v) = \\
&\quad= \frac{\sigma(u-v+w)\sigma(u+v+2w)}{\sigma(u-v)\sigma(u+v+w)}\times \\
&\quad\times\Bigg\{\sum_{\alpha_1,\alpha_2,\beta_1,\beta_2=2}^{n} R(u+v+w,\lambda-w\hat{\imath})^{k\ \ \beta_2}_{\alpha_2\,\beta_1} R(u-v,\lambda+w\hat{\jmath})^{\beta_1\,\alpha_1}_{j\ \ i}\times \\
&\qquad\times\mathcal{B}_{\beta_2}(\lambda+w(\hat{k}+\hat{\beta}_2-\hat{\imath}-\hat{1})|v)\Delta^{\alpha_2}_{\alpha_1}(\lambda+w(\hat{\jmath}-\hat{1})|u)\Bigg\} - \\
&\quad-\frac{\sigma(w)\sigma(2u+2w)}{\sigma(u-v)\sigma(2u+w)}\Bigg\{\sum_{\alpha,\beta=2}^{n}\frac{\sigma(u-v+\lambda_{1\alpha}-w)}{\sigma(\lambda_{1\alpha}-w)}R(2u+w,\lambda-w\hat{\imath})^{k\,\beta}_{\alpha\,i}\times \\
&\qquad\times\mathcal{B}_{\beta}(\lambda+w(\hat{k}+\hat{\beta}-\hat{\imath}-\hat{1})|u)\Delta^{\alpha}_{j}(\lambda+w(\hat{\jmath}-\hat{1})|v)\Bigg\} + \\
&\quad+\frac{\sigma(w)\sigma(2v)\sigma(2u+2w)}{\sigma(u+v+w)\sigma(2v+w)\sigma(2u+w)}\Bigg\{\sum_{\alpha=2}^{n}\frac{\sigma(u+v+\lambda_{1j})}{\sigma(\lambda_{1j}-w)}R(2u+w,\lambda-w\hat{\imath})^{k\,\alpha}_{j\,i}\times \\
&\qquad\times\mathcal{B}_{\alpha}(\lambda+w(\hat{k}+\hat{\alpha}-\hat{\imath}-\hat{1})|u)\times \\
&\qquad\mathcal{A}(\lambda+w(\hat{\jmath}-\hat{1})|v)\Bigg\},
\end{aligned}
\tag{4.29}
$$

(4.28)

$$
\begin{aligned}
&\mathcal{B}_i(\lambda+w(\hat{\imath}-\hat{1})|u)\mathcal{B}_j(\lambda+w(\hat{\imath}+\hat{\jmath}-2\hat{1})|v) = \\
&\quad=\sum_{\alpha,\beta=2}^{n} R(u-v,\lambda-2w\hat{1})^{\beta\,\alpha}_{j\,i}\,\mathcal{B}_\beta(\lambda+w(\hat{\beta}-\hat{1})|v)\,\mathcal{B}_\alpha(\lambda+w(\hat{\alpha}+\hat{\beta}-2\hat{1})|u)\,.
\end{aligned}
\tag{4.30}
$$

In order to apply the algebraic Bethe ansatz method, in addition to the relevant commutation relations (4.28)–(4.30), one needs to construct a pseudo-vacuum state (also called reference state) which is the common eigenstate of the operators $\mathcal{A}$, Δ_i^i and is annihilated by the operators $\mathcal{C}_i$. In contrast to the trigonometric and rational cases with *diagonal* $K^{\pm}(u)$ [18, 35], the usual highest-weight state

$$
\begin{pmatrix}1\\0\\\vdots\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}1\\0\\\vdots\end{pmatrix},
\tag{4.31}
$$

is no longer the pseudo-vacuum state. However, after the face-vertex transformations (4.18) and (4.19), the face type K-matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ *simultaneously* become diagonal. This suggests that one can translate the $\mathbb{Z}_n$ Belavin model with non-diagonal K-matrices into the corresponding SOS model with *diagonal* K-matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ given by (4.18)–(4.19). Then one can construct the pseudo-vacuum in the "face language" and use the algebraic Bethe ansatz method to diagonalize the transfer matrix [29].

Let us introduce the corresponding pseudo-vacuum state $|\Omega\rangle$

$$|\Omega\rangle = \phi_{\lambda-(N-1)w\hat{1},\lambda-Nw\hat{1}}(-z_1) \otimes \phi_{\lambda-(N-2)w\hat{1},\lambda-(N-1)w\hat{1}}(-z_2) \cdots \otimes \phi_{\lambda,\lambda-w\hat{1}}(-z_N) \,. \quad (4.32)$$

This state depends only on the boundary parameters $\{\lambda_i\}$ and the inhomogeneous parameters $\{z_j\}$, not on the boundary parameter ξ. We find that the pseudo-vacuum state given by (4.32) satisfies the following equations, as required,

$$\begin{aligned}
\mathcal{A}(\lambda - Nw\hat{1}|u)|\Omega\rangle &= k(\lambda|u;\xi)_1|\Omega\rangle \,, && (4.33)\\
\Delta^i_j(\lambda - Nw\hat{1}|u)|\Omega\rangle &= \delta^i_j f^{(1)}(u)\, k\left(\lambda|u + \frac{w}{2};\xi - \frac{w}{2}\right)_j \times \\
&\quad \times \left\{ \prod_{k=1}^{N} \frac{\sigma(u+z_k)\sigma(u-z_k)}{\sigma(u+z_k+w)\sigma(u-z_k+w)} \right\} |\Omega\rangle \,, \quad i,j = 2,\ldots,n \,, && (4.34)\\
\mathcal{C}_i(\lambda - Nw\hat{1}|u)|\Omega\rangle &= 0 \,, \quad i = 2,\ldots,n \,, && (4.35)\\
\mathcal{B}_i(\lambda - Nw\hat{1}|u)|\Omega\rangle &\neq 0 \,, \quad i = 2,\ldots,n \,. && (4.36)
\end{aligned}$$

Here $f^{(1)}(u)$ is given by

$$f^{(1)}(u) = \frac{\sigma(2u)\sigma(\lambda_1 + u + w + \xi)}{\sigma(2u+w)\sigma(\lambda_1 + u + \xi)} \,. \quad (4.37)$$

In order to apply the algebraic Bethe ansatz method to diagonalize the transfer matrix, we need to assume $N = n \times l$ with l being a positive integer [29]. For convenience, let us introduce a set of integers:

$$N_i = (n - i) \times l \,, \qquad i = 0, 1, \ldots, n-1 \,, \quad (4.38)$$

and $\frac{n(n-1)}{2} l$ complex parameters $\{v^{(i)}_k |\ k = 1, 2, \ldots, N_{i+1},\ i = 0, 1, \ldots, n-2\}$. As in the usual nested Bethe ansatz method [36, 37, 35, 38, 29], the parameters $\{v^{(i)}_k\}$ will be used to specify the eigenvectors of the corresponding reduced transfer matrices. They will be constrained later by the Bethe ansatz equations. For convenience, we adopt the following convention:

$$v_k = v^{(0)}_k \,, \qquad k = 1, 2, \ldots, N_1 \,. \quad (4.39)$$

We will seek the common eigenvectors (i.e. the so-called Bethe states) of the transfer matrix in the form

$$\begin{aligned}
|v_1, \ldots, v_{N_1}\rangle = &\sum_{i_1,\ldots,i_{N_1}=2}^{n} F^{i_1,i_2,\ldots,i_{N_1}} \mathcal{B}_{i_1}(\lambda + w(\hat{\imath}_1 - \hat{1})|v_1)\, \mathcal{B}_{i_2}(\lambda + w(\hat{\imath}_1 + \hat{\imath}_2 - 2\hat{1})|v_2) \times \\
&\times \cdots \mathcal{B}_{i_{N_1}}(\lambda + w\sum_{k=1}^{N_1} \hat{\imath}_k - wN_1\hat{1}|v_{N_1})|\Omega\rangle \,. \qquad (4.40)
\end{aligned}$$

The indices in the above equation should satisfy the following condition: the number of $i_k = j$, denoted by $\#(j)$, is l, i.e.

$$\#(j) = l \,, \qquad j = 2, \ldots, n \,. \quad (4.41)$$

With the help of (4.23), (4.26) and (4.27) we rewrite the transfer matrix (2.16) in terms of the operators $\mathcal{A}$ and Δ_i^i

$$\begin{aligned}
\tau(u) &= \sum_{\nu=1}^{n} \tilde{k}(\lambda|u)_\nu\, \mathcal{T}(\lambda|u)_\nu^\nu \\
&= \tilde{k}(\lambda|u)_1\, \mathcal{A}(\lambda|u) + \sum_{i=2}^{n} \tilde{k}(\lambda|u)_i\, \mathcal{T}(\lambda|u)_i^i \\
&= \tilde{k}(\lambda|u)_1\, \mathcal{A}(\lambda|u) + \sum_{i=2}^{n} \tilde{k}(\lambda|u)_i\, R(2u, \lambda + w\hat{1})_{1i}^{i1}\, \mathcal{A}(\lambda|u) \\
&\quad + \sum_{i=2}^{n} \tilde{k}(\lambda|u)_i \left(\mathcal{T}(\lambda|u)_i^i - R(2u, \lambda + w\hat{1})_{1i}^{i1} \mathcal{A}(\lambda|u)\right) \\
&= \sum_{i=1}^{n} \tilde{k}(\lambda|u)_i R(2u, \lambda + w\hat{1})_{1i}^{i1}\, \mathcal{A}(\lambda|u) + \\
&\quad + \sum_{i=2}^{n} \tilde{k}^{(1)}\left(\lambda|u + \frac{w}{2}\right)_i \frac{\sigma(\lambda_{i1} - w)}{\sigma(\lambda_{i1})} \left(\mathcal{T}(\lambda|u)_i^i - R(2u, \lambda + w\hat{1})_{1i}^{i1} \mathcal{A}(\lambda|u)\right) \\
&= \alpha^{(1)}(u)\, \mathcal{A}(\lambda|u) + \sum_{i=2}^{n} \tilde{k}^{(1)}(\lambda|u + \frac{w}{2})_i\, \Delta(\lambda|u)_i^i\,. \qquad (4.42)
\end{aligned}$$

Here we have used (4.27) and introduced the function $\alpha^{(1)}(u)$,

$$\alpha^{(1)}(u) = \sum_{i=1}^{n} \tilde{k}(\lambda|u)_i\, R(2u, \lambda + w\hat{1})_{1i}^{i1}\,, \qquad (4.43)$$

and the reduced K-matrix $\tilde{\mathcal{K}}^{(1)}(\lambda|u)$ with the elements given by

$$\tilde{\mathcal{K}}^{(1)}(\lambda|u)_i^j = \delta_i^j\, \tilde{k}^{(1)}(\lambda|u)_i\,, \qquad i, j = 2, \ldots, n\,, \qquad (4.44)$$

$$\tilde{k}^{(1)}(\lambda|u)_i = \left\{\prod_{k\neq i, k=2}^{n} \frac{\sigma(\lambda_{ik} - w)}{\sigma(\lambda_{ik})}\right\} \frac{\sigma(\lambda_i + \bar{\xi} + u + \frac{(n-1)w}{2})}{\sigma(\lambda_i + \bar{\xi} - u - \frac{(n-1)w}{2})}\,. \qquad (4.45)$$

To carry out the nested Bethe ansatz process [36, 37, 35, 38, 29] for the $\mathbb{Z}_n$ Belavin model with the generic open boundary conditions, one needs to introduce a set of reduced K-matrices $\tilde{\mathcal{K}}^{(m)}(\lambda|u)$ [29] which include the original one $\tilde{\mathcal{K}}(\lambda|u) = \tilde{\mathcal{K}}^{(0)}(\lambda|u)$ and the ones in (4.44) and (4.45):

$$\tilde{\mathcal{K}}^{(m)}(\lambda|u)_i^j = \delta_i^j\, \tilde{k}^{(m)}(\lambda|u)_i\,, \quad i, j = m+1, \ldots, n\,, \quad m = 0, \ldots, n-1\,, \qquad (4.46)$$

$$\tilde{k}^{(m)}(\lambda|u)_i = \left\{\prod_{k\neq i, k=m+1}^{n} \frac{\sigma(\lambda_{ik} - w)}{\sigma(\lambda_{ik})}\right\} \frac{\sigma(\lambda_i + \bar{\xi} + u + \frac{(n-m)w}{2})}{\sigma(\lambda_i + \bar{\xi} - u - \frac{(n-m)w}{2})}\,. \qquad (4.47)$$

Moreover we introduce a set of functions $\{\alpha^{(m)}(u)|m = 1, \ldots, n-1\}$ (including the one in (4.43)) related to the reduced K-matrices $\tilde{\mathcal{K}}^{(m)}(\lambda|u)$

$$\alpha^{(m)}(u) = \sum_{i=m}^{n} R(2u, \lambda + w\hat{m})_{mi}^{im}\, \tilde{k}^{(m-1)}(\lambda|u)_i\,, \quad m = 1, \ldots, n\,. \qquad (4.48)$$

Carrying out the nested Bethe ansatz, we find [29] that with the coefficients $F^{i_1,i_2,\ldots,i_{N_1}}$ in (4.40) properly chosen, the Bethe state $|v_1,\ldots,v_{N_1}\rangle$ is the eigenstate of the transfer matrix (2.16),

$$\tau(u)|v_1,\ldots,v_{N_1}\rangle = \Lambda(u;\xi,\{v_k\})|v_1,\ldots,v_{N_1}\rangle\,, \tag{4.49}$$

with eigenvalues given by

$$\begin{aligned}\Lambda(u;\xi,\{v_k\}) \;=\; &\alpha^{(1)}(u)\frac{\sigma(\lambda_1+\xi-u)}{\sigma(\lambda_1+\xi+u)}\prod_{k=1}^{N_1}\frac{\sigma(u+v_k)\sigma(u-v_k-w)}{\sigma(u+v_k+w)\sigma(u-v_k)}+\\ &+\frac{\sigma(2u)\sigma(\lambda_1+u+w+\xi)}{\sigma(2u+w)\sigma(\lambda_1+u+\xi)}\prod_{k=1}^{N_1}\frac{\sigma(u-v_k+w)\sigma(u+v_k+2w)}{\sigma(u-v_k)\sigma(u+v_k+w)}\times\\ &\times\prod_{k=1}^{N_0}\frac{\sigma(u+z_k)\sigma(u-z_k)}{\sigma(u+z_k+w)\sigma(u-z_k+w)}\Lambda^{(1)}\left(u+\frac{w}{2};\xi-\frac{w}{2},\{v^{(1)}\}\right).\end{aligned} \tag{4.50}$$

The eigenvalues $\{\Lambda^{(i)}(u;\xi,\{v_k^{(i)}\})\}$ (with the original one $\Lambda(u;\xi,\{v_k\})=\Lambda^{(0)}(u;\xi,\{v_k^{(0)}\})$) of the reduced transfer matrices are given by the following recurrence relations

$$\begin{aligned}\Lambda^{(i)}(u;\xi^{(i)},\{v_k^{(i)}\}) \;=\; &\alpha^{(i+1)}(u)\frac{\sigma(\lambda_{i+1}+\xi^{(i)}-u)}{\sigma(\lambda_{i+1}+\xi^{(i)}+u)}\prod_{k=1}^{N_{i+1}}\frac{\sigma(u+v_k^{(i)})\sigma(u-v_k^{(i)}-w)}{\sigma(u+v_k^{(i)}+w)\sigma(u-v_k^{(i)})}+\\ &+\frac{\sigma(2u)\sigma(\lambda_{i+1}+u+w+\xi^{(i)})}{\sigma(2u+w)\sigma(\lambda_{i+1}+u+\xi^{(i)})}\prod_{k=1}^{N_{i+1}}\frac{\sigma(u-v_k^{(i)}+w)\sigma(u+v_k^{(i)}+2w)}{\sigma(u-v_k^{(i)})\sigma(u+v_k^{(i)}+w)}\times\\ &\times\prod_{k=1}^{N_i}\frac{\sigma(u+z_k^{(i)})\sigma(u-z_k^{(i)})}{\sigma(u+z_k^{(i)}+w)\sigma(u-z_k^{(i)}+w)}\Lambda^{(i+1)}\left(u+\frac{w}{2};\xi^{(i)}-\frac{w}{2},\{v^{(i+1)}\}\right),\\ &\qquad\qquad i=1,\ldots,n-2\,,\end{aligned} \tag{4.51}$$

$$\Lambda^{(n-1)}(u;\xi^{(n-1)}) \;=\; \frac{\sigma(\lambda_n+\bar{\xi}+u+\frac{w}{2})\sigma(\lambda_n+\xi^{(n-1)}-u)}{\sigma(\lambda_n+\bar{\xi}-u-\frac{w}{2})\sigma(\lambda_n+\xi^{(n-1)}+u)}\,. \tag{4.52}$$

The reduced boundary parameters $\{\xi^{(i)}\}$ and inhomogeneous parameters $\{z_k^{(i)}\}$ are given by

$$\xi^{(i+1)}=\xi^{(i)}-\frac{w}{2}\,,\qquad z_k^{(i+1)}=v_k^{(i)}+\frac{w}{2}\,,\qquad i=0,\ldots,n-2\,. \tag{4.53}$$

Here we have adopted the convention: $\xi=\xi^{(0)}$, $z_k^{(0)}=z_k$. The complex parameters $\{v_k^{(i)}\}$ satisfy the following Bethe ansatz equations:

$$\begin{aligned}&\alpha^{(1)}(v_s)\frac{\sigma(\lambda_1+\xi-v_s)\sigma(2v_s+w)}{\sigma(\lambda_1+\xi+v_s+w)\sigma(2v_s+2w)}\prod_{k\neq s,k=1}^{N_1}\frac{\sigma(v_s+v_k)\sigma(v_s-v_k-w)}{\sigma(v_s+v_k+2w)\sigma(v_s-v_k+w)}=\\ &\quad=\prod_{k=1}^{N}\frac{\sigma(v_s+z_k)\sigma(v_s-z_k)}{\sigma(v_s+z_k+w)\sigma(v_s-z_k+w)}\Lambda^{(1)}\left(v_s+\frac{w}{2};\xi-\frac{w}{2},\{v_k^{(1)}\}\right),\end{aligned} \tag{4.54}$$

$$\alpha^{(i+1)}(v_s^{(i)})\frac{\sigma(\lambda_{i+1}+\xi^{(i)}-v_s^{(i)})\sigma(2v_s^{(i)}+w)}{\sigma(\lambda_{i+1}+\xi^{(i)}+v_s^{(i)}+w)\sigma(2v_s^{(i)}+2w)}\times$$

$$\times \prod_{k\neq s,k=1}^{N_{i+1}} \frac{\sigma(v_s^{(i)}+v_k^{(i)})\sigma(v_s^{(i)}-v_k^{(i)}-w)}{\sigma(v_s^{(i)}+v_k^{(i)}+2w)\sigma(v_s^{(i)}-v_k^{(i)}+w)} =$$

$$= \prod_{k=1}^{N_i} \frac{\sigma(v_s^{(i)}+z_k^{(i)})\sigma(v_s^{(i)}-z_k^{(i)})}{\sigma(v_s^{(i)}+z_k^{(i)}+w)\sigma(v_s^{(i)}-z_k^{(i)}+w)}\,\Lambda^{(i+1)}\left(v_s^{(i)}+\frac{w}{2};\xi^{(i)}-\frac{w}{2},\{v_k^{(i+1)}\}\right),$$

$$i=1,\ldots,n-2. \tag{4.55}$$

4.3 Eigenvalues and associated Bethe ansatz equations

The relation (3.8) between $\{H_j\}$ and $\{\tau(z_j)\}$ and the fact that the first term on the r.h.s. of (3.7) is a c-number enable us to extract the eigenstates of the elliptic Gaudin operators and the corresponding eigenvalues from the results obtained in the previous subsection.

Introduce the functions $\{\beta^{(i)}(u,w)\}$

$$\beta^{(i+1)}(u,w) \equiv \beta^{(i+1)}(u) = \alpha^{(i+1)}(u)\frac{\sigma(\lambda_{i+1}+\xi-u-\frac{i}{2}w)}{\sigma(\lambda_{i+1}+\xi+u+w-\frac{i}{2}w)}, \quad i=0,\ldots,n-2. \tag{4.56}$$

Then by (4.51), (4.52) and (4.53), the Bethe ansatz equations (4.54) and (4.55) become, respectively,

$$\beta^{(i+1)}(v_s^{(i)})\frac{\sigma(2v_s^{(i)}+w)}{\sigma(2v_s^{(i)}+2w)}\prod_{k\neq s,k=1}^{N_{i+1}}\frac{\sigma(v_s^{(i)}+v_k^{(i)})\sigma(v_s^{(i)}-v_k^{(i)}-w)}{\sigma(v_s^{(i)}+v_k^{(i)}+2w)\sigma(v_s^{(i)}-v_k^{(i)}+w)} =$$

$$= \beta^{(i+2)}\left(v_s^{(i)}+\frac{w}{2}\right)\prod_{k=1}^{N_i}\frac{\sigma(v_s^{(i)}+v_k^{(i-1)}+\frac{w}{2})\sigma(v_s^{(i)}-v_k^{(i-1)}-\frac{w}{2})}{\sigma(v_s^{(i)}+v_k^{(i-1)}+\frac{3w}{2})\sigma(v_s^{(i)}-v_k^{(i-1)}+\frac{w}{2})}\times$$

$$\times \prod_{k=1}^{N_{i+2}}\frac{\sigma(v_s^{(i)}+v_k^{(i+1)}+\frac{w}{2})\sigma(v_s^{(i)}-v_k^{(i+1)}-\frac{w}{2})}{\sigma(v_s^{(i)}+v_k^{(i+1)}+\frac{3w}{2})\sigma(v_s^{(i)}-v_k^{(i+1)}+\frac{w}{2})}, \qquad i=0,\ldots,n-3, \tag{4.57}$$

$$\beta^{(n-1)}(v_s^{(n-2)})\frac{\sigma(2v_s^{(n-2)}+w)}{\sigma(2v_s^{(n-2)}+2w)}\prod_{k\neq s,k=1}^{N_{n-1}}\frac{\sigma(v_s^{(n-2)}+v_k^{(n-2)})\sigma(v_s^{(n-2)}-v_k^{(n-2)}-w)}{\sigma(v_s^{(n-2)}+v_k^{(n-2)}+2w)\sigma(v_s^{(n-2)}-v_k^{(n-2)}+w)} =$$

$$= \frac{\sigma(\lambda_n+\xi+v_s^{(n-2)}+w)\sigma(\lambda_n+\xi^{(n-1)}-v_s^{(n-2)}-\frac{w}{2})}{\sigma(\lambda_n+\xi-v_s^{(n-2)}-w)\sigma(\lambda_n+\xi^{(n-1)}+v_s^{(n-2)}+\frac{w}{2})}\times$$

$$\times \prod_{k=1}^{N_{n-2}}\frac{\sigma(v_s^{(n-2)}+v_k^{(n-3)}+\frac{w}{2})\sigma(v_s^{(n-2)}-v_k^{(n-3)}-\frac{w}{2})}{\sigma(v_s^{(n-2)}+v_k^{(n-3)}+\frac{3w}{2})\sigma(v_s^{(n-2)}-v_k^{(n-3)}+\frac{w}{2})}. \tag{4.58}$$

Here we have used the convention: $v_k^{(-1)} = z_k$, $k=1,\ldots,N$. The quasi-classical property (4.7) of $R(u,m)$, (4.48) and (4.56) lead to the following relations

$$\beta^{(i+1)}(u,0)=1\,, \quad \frac{\partial}{\partial u}\beta^{(i+1)}(u,0)=0\,, \quad i=0,\ldots,n-2. \tag{4.59}$$

Then, one may introduce functions $\{\gamma^{(i+1)}(u)\}$ associated with $\{\beta^{(i+1)}(u,w)\}$

$$\gamma^{(i+1)}(u) = \frac{\partial}{\partial w}\beta^{(i+1)}(u,w)|_{w=0}\,, \quad i=0,\ldots,n-2. \tag{4.60}$$

Using (4.59), we can extract the eigenvalues h_j (resp. the corresponding Bethe ansatz equations) of the Gaudin operators H_j (3.6) from the expansion around $w = 0$ for the first order of w of the eigenvalues (4.50) of the transfer matrix $\tau(u = z_j)$ (resp. the Bethe ansatz equations (4.57) and (4.58)). Finally, the eigenvalues of the $\mathbb{Z}_n$ elliptic Gaudin operators are

$$h_j = \gamma^{(1)}(z_j) + \zeta(\lambda_1 + \xi + z_j) - \sum_{k=1}^{N_1} \{\zeta(z_j + x_k) + \zeta(z_j - x_k)\}, \tag{4.61}$$

where ζ-function is defined in (2.3). The $\frac{n(n-1)l}{2}$ parameters $\{x_k^{(i)}|\ k = 1, 2, \ldots, N_{i+1},\ i = 0, 1, \ldots, n-2\}$ (including x_k as $x_k = x_k^{(0)}$, $k = 1, \ldots, N_1$) are determined by the following associated Bethe ansatz equations

$$\begin{aligned}
&\gamma^{(i+1)}(x_s^{(i)}) - \zeta(2x_s^{(i)}) - 2\sum_{k\neq s,k=1}^{N_{i+1}} \left\{\zeta(x_s^{(i)} + x_k^{(i)}) + \zeta(x_s^{(i)} - x_k^{(i)})\right\} = \\
&\quad = \gamma^{(i+2)}(x_k^{(i)}) - \sum_{k=1}^{N_i} \left\{\zeta(x_s^{(i)} + x_k^{(i-1)}) + \zeta(x_s^{(i)} - x_k^{(i-1)})\right\} - \\
&\qquad - \sum_{k=1}^{N_{i+2}} \left\{\zeta(x_s^{(i)} + x_k^{(i+1)}) + \zeta(x_s^{(i)} - x_k^{(i+1)})\right\}, \quad i = 0, \ldots, n-3,
\end{aligned} \tag{4.62}$$

$$\begin{aligned}
&\gamma^{(n-1)}(x_s^{(n-2)}) - \zeta(2x_s^{(n-2)}) - 2\sum_{k\neq s,k=1}^{N_{n-1}} \left\{\zeta(x_s^{(n-2)} + x_k^{(n-2)}) + \zeta(x_s^{(n-2)} - x_k^{(n-2)})\right\} = \\
&\quad = \frac{n}{2}\zeta(\lambda_n + \xi + x_s^{(n-2)}) + \frac{2-n}{2}\zeta(\lambda_n + \xi - x_s^{(n-2)}) - \\
&\qquad - \sum_{k=1}^{N_{n-2}} \left\{\zeta(x_s^{(n-2)} + x_k^{(n-3)}) + \zeta(x_s^{(n-2)} - x_k^{(n-3)})\right\}.
\end{aligned} \tag{4.63}$$

Here we have used the convention: $x_k^{(-1)} = z_k$, $k = 1, \ldots, N$ in (4.62).

5. Conclusions

We have studied the $\mathbb{Z}_n$ elliptic Gaudin model with generic boundaries specified by the non-diagonal K-matrices $K^{\pm}(u)$, (3.1) and (3.2). In addition to the inhomogeneous parameters $\{z_j\}$, the associated Gaudin operators $\{H_j\}$, (3.6), have $n + 1$ free parameters $\{\lambda_i\}$ and ξ. As seen from section 4, although the "vertex type" K-matrices $K^{\pm}(u)$ (3.1) and (3.2) are *non-diagonal*, the compositions, (4.18) and (4.19), lead to the *diagonal* "face-type" K-matrices after the face-vertex transformation. This enables us to successfully construct the corresponding pseudo-vacuum state $|\Omega\rangle$ (4.32) and apply the algebraic Bethe ansatz method to diagonalize the transfer matrix $\tau(u)$ of the inhomogeneous $\mathbb{Z}_n$ Belavin model with generic open boundaries. Furthermore, we have exactly diagonalized the generalized Gaudin operators $\{H_j\}$, and derived their eigenvalues (4.61) as well as the associated Bethe ansatz equations (4.62) and (4.63). Taking the scaling limit [39] of our general results for

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the special $n = 2$ case, we recover the results obtained in [25] for the XXZ Gaudin model with generic open boundaries.

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References

[1] M. Gaudin, *Diagonalisation d'une classe d'Hamiltoniens de spin*, *J. Phys.* (Paris) **37** (1976) 1087.

[2] J. Bardeen, L.N. Cooper and J.R. Schrieffer, *Theory of superconductivity*, *Phys. Rev.* **108** (1957) 1175.

[3] M.C. Cambiaggio, A.M.F. Rivas and M. Saraceno, *Integrability of the pairing Hamiltonian*, *Nucl. Phys.* **A 624** (1997) 157

[4] L. Amico, A. Di Lorenzo and A. Osterloh, *Integrable models for confined fermions: applications to metallic grains*, *Phys. Rev. Lett.* **86** (2001) 5759 [cond-mat/0105537]; *Integrable models for confined fermions: applications to metallic grains*, *Phys. Rev. Lett.* **86** (2001) 5759 [cond-mat/0105537].

[5] J. von Delft and D.C. Ralph, *Spectroscopy of discrete energy levels in ultrasmall metallic grains*, *Phys. Rept.* **345** (2001) 61.

[6] D.J. Dean and M. Hjorth-Jensen, *Pairing in nuclear systems: from neutron stars to finite nuclei*, *Rev. Mod. Phys.* **75** (2003) 607 [nucl-th/0210033].

[7] J. Dukelsky, S. Pittel and G. Sierra, *Exactly solvable richardson-gaudin models for many-body quantum systems*, nucl-th/0405011.

[8] F. Iachelo, *Superconductivity in finite Fermi systems*, *Nucl. Phys.* **A 570** (1994) 145.

[9] D.H. Rischke and R.D. Pisarski, *Color superconductivity in cold, dense quark matter*, nucl-th/0004016.

[10] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087].

[11] H.M. Babujian, *Off-shell Bethe ansatz equations and N-point correlators in* SU(2) *WZNW theory*, *J. Phys.* **A 26** (1993) 6981.

[12] T. Hasegawa, K. Hikami and M. Wadati, *Integral solution of trigonometric Knizhnik-Zamolodchikov equation; off-shell Bethe ansatz method*, *J. Phys. Soc. Jpn.* **63** (1994) 2895.

[13] B. Feigin, E. Frenkel and N. Reshetikhin, *Gaudin model, Bethe ansatz and correlation functions at the critical level*, *Commun. Math. Phys.* **166** (1994) 27.

[14] K. Hikami, *Gaudin magnet with boundary and generalized Knizhnik-Zamolodchikov equation*, *J. Phys.* **A 28** (1995) 4997.

[15] M.D. Gould, Y.-Z. Zhang and S.-Y. Zhao, *Elliptic Gaudin models and elliptic KZ equations*, *Nucl. Phys.* **B 630** (2002) 492 [nlin.si/0110038].

[16] K. Hikami, P.P. Kulish and M. Wadati, *Construction of integrable spin systems with long-range interactions*, *J. Phys. Soc. Jpn.* **61** (1992) 3071.

[17] E.K. Sklyanin and T. Takebe, *Algebraic Bethe ansatz for the XYZ Gaudin model*, *Phys. Lett.* **A 219** (1996) 217.

[18] E.K. Sklyanin, *Boundary conditions for integrable quantum systems*, *J. Phys.* **A 21** (1988) 2375.

[19] A. Di Lorenzo, L. Amico, K. Hikami, A. Osterloh and G. Giaquinta, *Quasi-classical descendants of disordered vertex models with boundaries*, *Nucl. Phys.* **B 644** (2002) 409 [cond-mat/0206521].

[20] J. Dukelsky, C. Esebbag and P. Schuck, *Class of exactly solvable pairing models*, *Phys. Rev. Lett.* **87** (2001) 66403.

[21] H.Q. Zhou, J.R. Links, R. Mackenzie and M.D. Gould, *Superconducting correlations in metallic nanoparticles: exact solution of the BCS model by the algebraic Bethe ansatz*, *Phys. Rev.* **B 65** (2002) 060502.

[22] J. von Delft and R. Poghossian, *Algebraic Bethe ansatz for a discrete-state BCS pairing model*, *Phys. Rev.* **B 66** (2002) 134502.

[23] H.J. de Vega and A. Gonzalez-Ruiz, *Boundary K-matrices for the six-vertex and $n(2n-1)$ A_{n-1} vertex models*, *J. Phys.* **A 26** (1993) L519.

[24] S. Ghoshal and A.B. Zamolodchikov, *Boundary s matrix and boundary state in two-dimensional integrable quantum field theory*, *Int. J. Mod. Phys.* **A 9** (1994) 3841 [hep-th/9306002].

[25] W.-L. Yang, Y.-Z. Zhang, M. Gould, *Exact solution of the XXZ Gaudin model with generic open boundaries*, *Nucl. Phys.* **B 698** (2004) 503

[26] H. Fan, B.Y. Hou, G.L. Li and K.J. Shi, *A new solution to the reflection equation for the Z_n symmetric Belavin model*, *Phys. Lett.* **A 250** (1998) 79.

[27] W.-L. Yang and R. Sasaki, *Solution of the dual reflection equation for $A^{(1)}_{n-1}$ SOS model*, hep-th/0308118.

[28] A.A. Belavin, *Dynamical symmetry of integrable quantum systems*, *Nucl. Phys.* **B 180** (1981) 189.

[29] W.L. Yang and R. Sasaki, *Exact solution of z_n belavin model with open boundary condition*, *Nucl. Phys.* **B 679** (2004) 495 [hep-th/0308127].

[30] E.T. Whittaker and G.N. Watson, *A course of modern analysis: 4th edn.*, Cambridge University Press, Cambridge 2002.

[31] M.P. Richey and C.A. Tracy, *Z_n Baxter model: symmetries and the Belavin parametrization*, *J. Stat. Phys.* **42** (1986) 311.

[32] B.Y. Hou and W.-L. Yang, *The non-dynamical r-matrix structure for the elliptic A_{n-1} Calogero-Moser model*, *J. Phys.* **A32** (1999) 1475.

[33] V.E. Korepin, N.M. Bogoliubov and A.G. Izergin, *Quantum inverse scattering method and correlation function*, Cambridge Univesity Press, Cambridge 1993.

[34] M. Jimbo, T. Miwa and M. Okado, *Solvable lattice models whose states are dominant integral weights of*$A^{(1)}_{n-1}$, *Lett. Math. Phys.* **14** (1987) 123; *Local state probabilities of solvable lattice modles:* $A^{(1)}_{n-1}$ *family*, *Nucl. Phys.* **B 300** (1988) 74.

[35] H.J. de Vega and A. Gonzalez-Ruiz, *Exact solution of the* $SU_q(n)$ *invariant quantum spin chains*, *Nucl. Phys.* **B 417** (1994) 553 [hep-th/9309022].

[36] O. Babelon, H.J. de Vega and C.M. Viallet, *Exact solution of the* $Z(N+1)xZ(N+1)$ *symmetric generalization of the XXZ model*, *Nucl. Phys.* **B 200** (1982) 266.

[37] C.L. Schultz, *Eigenvectors of the multicomponent generalization of the six vertex model*, *Physica* **A122** (1983) 71.

[38] B.-y. Hou, R. Sasaki and W.-L. Yang, *Algebraic bethe ansatz for the elliptic quantum group* $e_{\tau,\eta}(sl_n)$ *and its applications*, *Nucl. Phys.* **B 663** (2003) 467 [hep-th/0303077].

[39] W.-L. Yang and Y. Zhen, *Modular transformation and the twist between trigonometric limits of* $sl(n)$ *elliptic R-matrix*, *Commun. Theor. Phys.* **36** (2001) 131, math-ph/0103028.